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Chapter 4

THE INITIAL-BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS IN DOMAINS WITH CONICAL POINTS

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Abstract

In this paper we consider a initial-boundary value problem for parabolic equation in a domain with conical points. We establish the weak well-posedness of the problem and the regularity of the solution. We also receive asymptotic formulas for the solution near the conical points.

Keywords: parabolic equation, initial-boundary value problem, nonsmooth domains, generalized solutions, regularity, asymptotics.

2000 MSC: 35D05, 35D10, 35K30, 35B40.

1. Introduction

Elliptic boundary value problems in domains with point singularities were thoroughly investigated (see, e.g, [4,9] and the extensive bibliography therein). In this paper we deal with a initial-boundary value problem for parabolic equation in a domain with conical points. We investigate the unique solvability of the problem and asymptotics for solution near singularities of the boundary of the domain.

Initial-boundary value problems for parabolic equations are studied widely on various aspects with differential approaches. For example, for parabolic equations of second order

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in a smooth domains, the solvability of the problems and the regularity of the solutions were established in both Hölder and Sobolev spaces in [15] by the method in which a regularizer was constructed and exact estimates of solutions in terms of the data were dealt with. Such ideas were also used in [6] with some modifications for the case of domains with edges. For the equation considered in [21], whose coefficients are independent of the time variable, the Fourier transform was used to reduce the problem to an elliptic one with a parameter. In the present paper, for a general higher order linear parabolic equation with coefficients depending on both spatial and time variable in a domain containing conical points we modify the approach suggested in [8, 11, 12]. First, in Section 3., we study the unique solvability of the problem and the regularity with respect to the time variable for the generalized solution by Galerkin's approximate method. By modifying the arguments used in [11, 12], we can weaken the restrictions on the data at the initial time $t = 0$ imposed therein. These results are essential contained in the authors' paper [13]. After that, in Section 4., we deal with asymptotics of the solution of the problem near singularities of the boundary. To this end, we take the term containing the derivative in time of the unknown function to the right-hand side of the equation such that the problem can be considered as an elliptic one depending parameter t. With some assumptions on the smoothness with respect to the variable t of eigenvalues, eigenvectors and generalized eigenvectors of the operator pencil generated by the problem we apply the results on elliptic boundary value problems in domains with conical points and our previous results on the regularity of the solutions to receive asymptotic formulas for the solution near singularities of the boundary as a sum of a linear combination of special singular functions and a regular function in which this functions and the coefficients of the linear combination are regular with respect to the variable t. Finally, in Section 5., with the help of the results on the analytic perturbation theory of linear operators ([7,20]) and the method of linearization of polynomial operator pencils ([19]) we deal with the operator pencil generated by the problem. Our purpose here is to give a sufficient condition for the assumptions mentioned above to be valid.

2. Notation and Formulation of the Problem

Let G be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$ with the boundary ∂G. We suppose that $S = \partial G \setminus \{0\}$ is a smooth manifold and G in a neighborhood U of the origin 0 coincides with the cone $K = \{x : x/|x| \in \Omega\}$, where Ω is a smooth domain with the boundary Γ on the unit sphere S^{n-1} in $\mathbb{R}^n$. Let T be a positive real number or $T = +\infty$. For $0 \leqslant t \leqslant T$, we set $G_t = G \times (0,t), \Omega_t = \Omega \times (0,t), S_t = S \times [0,t], \Gamma_t = \Gamma \times [0,t]$. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, set $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial_x^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$. For a nonnegative integer k we write u_{t^k} instead of $\partial_t^k u$.

Let us introduce some functional space used in this paper. Let l be a nonnegative integer. We denote by $W_2^l(G)$ the usual Sobolev space of functions defined in G with the norm

$$\|u\|_{W_2^l(G)} = \left(\int_G \sum_{|\alpha| \leqslant m} |\partial_x^\alpha u|^2 dx \right)^{\frac{1}{2}},$$

and by $W_2^{l-\frac{1}{2}}(S)$ the space of traces of functions from $W_2^l(G)$ on S with the norm

$$\|u\|_{W_2^{l-\frac{1}{2}}(S)} = \inf\left\{\|v\|_{W_2^l(G)} : v \in W_2^l(G), v|_S = u\right\}.$$

We denote by $W_{2,loc}^l(G)$ the space of all functions $u(x)$ defined in G such that they and all their derivatives up to order l are square integrable on every compact subset of G, and by $\mathcal{W}_{2,loc}^{2lm,l}(G_T)$ the space of all functions $u(x,t)$ defined in G_T with the derivatives

$$\partial_x^\alpha u_{t^k}, |\alpha| + 2mk \leqslant 2lm,$$

being square integrable on every compact subset of G_T.

We define the weighted Sobolev space $V_{2,\gamma}^l(K)$ ($\gamma \in \mathbb{R}$) as the closure of $C_0^\infty(\overline{K}\setminus\{0\})$ with respect to the norm

$$\|u\|_{V_{2,\gamma}^l(K)} = \left(\sum_{|\alpha|\leqslant l}\int_K r^{2(\gamma+|\alpha|-l)}|\partial_x^\alpha u|^2 dx\right)^{\frac{1}{2}}, \tag{2.1}$$

where $r = |x| = \left(\sum_{k=1}^n x_k^2\right)^{\frac{1}{2}}$. If $l \geqslant 1$, then $V_\gamma^{l-\frac{1}{2}}(\partial K)$ denote the space consisting of traces of functions from $V_{2,\gamma}^l(K)$ on the boundary ∂K with the norm

$$\|u\|_{V_\gamma^{l-\frac{1}{2}}(\partial K)} = \inf\left\{\|v\|_{V_{2,\gamma}^l(K)} : v \in V_{2,\gamma}^l(K), v|_{\partial K} = u\right\}. \tag{2.2}$$

The weighted spaces $V_{2,\gamma}^l(G)$, $V_\gamma^{l-\frac{1}{2}}(S)$ are defined similarly as in (2.1), (2.2) with $K, \partial K$ replaced by G, S, respectively.

It is obvious from the definition that the space $V_{2,\gamma+k}^{l+k}(G)$ is continuously imbedded into the space $V_{2,\gamma}^l(G)$ for an arbitrary nonnegative integer k. An analogous assertion holds for the space $V_{2,\gamma}^{l-\frac{1}{2}}(\partial G)$.

We define $W_{2,\gamma}^l(G)$ ($\gamma \in \mathbb{R}$) as the set of all functions in G such that $r^\gamma \partial_x^\alpha u \in L_2(G)$ for $|\alpha| \leq l$ with the norm

$$\|u\|_{W_{2,\gamma}^l(G)} = \left(\sum_{|\alpha|\leqslant l}\int_G r^{2\gamma}|\partial_x^\alpha u|^2 dx\right)^{\frac{1}{2}}.$$

If $l \geqslant 1$, then $W_{2,\gamma}^{l-\frac{1}{2}}(S)$ denote the space consisting of traces of functions from respective space $W_{2,\gamma}^l(G)$ on the boundary S with the norm

$$\|u\|_{W_{2,\gamma}^{l-\frac{1}{2}}(S)} = \inf\left\{\|v\|_{W_{2,\gamma}^l(G)} : v \in W_{2,\gamma}^l(G), v|_S = u\right\}.$$

Let X, Y be Banach spaces. Denote by $\mathcal{B}(X,Y)$ the set of all continuous linear operators from X into Y. If $Y = X$, we write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,Y)$. We denote by $L_2((0,T);X)$ the space consisting of all measurable functions $u : (0,T) \to X$ with the norm

$$\|u\|_{L_2((0,T);X)} = \left(\int_0^T \|u(t)\|_X^2 dt\right)^{\frac{1}{2}},$$